THE
MATHEMATICAL GAZETTE.

EDITED BY
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WITH THE CO-OPERATION OF
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LOCAL BRANCHES.

THE FIRST MEETING OF LONDON BRANCH. DISCUSSION ON THE BOARD OF EDUCATION CIRCULAR (GEOMETRY).

THE first meeting of the London Branch was held on Jan. 29th at the Polytechnic, Regent Street. There was a large attendance, about 220 being present. The first part of the meeting was devoted to the discussion of the constitution of the Branch and the election of officers. The following were elected officers for 1910: *Chairman*, Mr. C. S. Jackson: *Vice-Chairmen*, Dr. T. P. Nunn and F. J. W. Whipple; *Treasurer*, Mr. W. M. Roberts; *Hon. Secretaries*, Miss H. K. Scott and Mr. P. Abbott; *Committee*, Misses A. E. Bennett, B. M. Cave, B. Hewett, S. Hunt, M. M'Afee, H. M. Sheldon, and C. M. Waters; Dr. Macaulay, J. V. H. Coates, G. F. Daniell, M. Gheury, W. Knowles, W. E. Paterson, Howard Smart, and J. Stanton Wise. The chief business of the meeting was a discussion on the Board of Education circular so far as it referred to the teaching of Geometry. This was opened by Mr. A. W. Siddons, and an interesting discussion followed.

THE DISCUSSION.

Mr. A. W. Siddons, of Harrow School, pointed out that the circular said the teaching of geometry had improved, but he added that there would have been more improvement if there had been more discussion of new methods, and meetings such as that would do much by encouraging the exchange of ideas. He went on to speak of the aims of geometrical teaching, which he considered to be space-knowledge, and logical power. Though he regarded the latter as the more important, he pointed out the absurdity of beginning logical geometry before the child had a good deal of geometrical knowledge. He drew attention to

various points in the first part of the circular: the excellent treatment suggested for the fundamental concepts, the remarks on definitions, its condemnation of aimless drawing and teaching at this stage the geometrical constructions such as that for bisecting an angle.

Mr. Siddons went on to consider what should follow this first stage. Should the child be asked to argue inductively or deductively, or both? Certainly inductively at first; children naturally argue inductively. "Euclid did not write for children," and it is a mistake to expect them to adopt the deductive method easily—deductive power can only be developed slowly, and by arguing about familiar facts; hence the importance of treating the fundamental facts in the way suggested in the circular, and of postponing the logical proofs of them: further than that the fundamental facts are not the easiest to treat logically, and the proofs of them are unlike the proofs of other geometrical propositions, so that a knowledge of the proofs is of no real assistance in learning to argue logically. Here Mr. Siddons digressed on the method of arriving at the fundamental facts, the importance of children stating the facts and becoming thoroughly familiar with them by doing numerical exercises and plenty of easy riders.

The next question that arose was, should they go back after this stage and prove the fundamental facts, or go on? He advocated the latter course, but pointed out the importance of having these facts clearly stated, and of having it made clear that their truth was being assumed, and that the truth of future propositions would be dependent on these assumptions.

With regard to the third stage, he felt the line followed should still be that of all scientific research, viz. (i) experiment, (ii) a statement of any facts observed, (iii) experimental verification of the facts stated, (followed throughout this stage of course by) (iv) logical proofs of the facts. He also drew attention to the importance the circular laid on points such as studying propositions in groups, the possibility of other logical orders, the effect of gradual modification of figures, etc.

Finally, he pointed out that the scheme suggested by the circular was workable at once without undue effort on the part of teachers, and without any special qualifications beyond what they already possessed.

Miss E. Home: I shall be interested to hear the opinions of those present on that part of the memorandum which deals with what is called the first stage of geometry. At the bottom of page 1 it is stated that the chief aim of this part of the work is to give a clear idea of *geometrical concepts*.

I do not quite agree with this, for it seems to me that the right way to begin any subject with a child is to give it a bird's-eye

view of the thing as a whole, not appealing to his *intellect*, but to his *imagination*.

It is axiomatic with those who have to do with the education of the young, that since in children the *senses* are much more developed than the *intellect*, it is right to train them at first on lines which appeal more to the senses than to the intellect. Thus in the kindergarten, the teacher gives plenty of brushwork, singing, and conversational French, and does not give the children books and theories.

Again, in the teaching of *botany*, we do not begin by showing the children the stalk, petals, and leaves of a flower, but they first see the flower growing, then attend to the "concepts of the subject."

Similarly, in teaching *music*, we do not begin by pulling the modern scale to pieces, and discoursing on the various intervals of which it consists, but we let the children hear plenty of tunes, and then let them realise how these are constructed.

Hence it would seem to be a wise plan to begin the study of geometry by reminding the children about the curves they have seen in nature before limiting the discussion to the straight line and circle, even though the conic sections be added later. The study of curves will assume a really vital interest to a child who has realised the curves in a flower, in a wave, in a spiral of smoke, or in the path of an animal pursuing another.

These considerations will stimulate him to wish to investigate some of these curves for himself. He will then try to invent curves of his own, with but a small degree of success at first—as the ideas of law and data will not be familiar—and he will at first select unsuitable data and will apply his own law inaccurately.

But as he gains in experience he will relate these ideas with some of the forces underlying and running through everything; in other words, he will begin to relate mathematics with other rhythmic forces.

It may be objected to this method of beginning the study of geometry that the results achieved are not sufficiently "definite." But I would ask those who see it in this light whether the results on botanical or musical work, of letting a child see a living flower and hear a living tune before dissecting them, appear any more definite? And yet we are agreed that there is but one way to make these subjects vital to a child. These suggestions as to a similar method of beginning geometry would therefore not appear to be heretical.

Miss E. R. Gwatkin: I should like first of all to endorse very heartily the commendation given in the circular to rapid work. We are often told that at the beginning of a subject it is necessary to go very slowly in order to let the new ideas sink in.

I agree that it may be a good plan to repeat oneself a great deal, and to cover very little ground for the first few weeks, but some teachers seem to me to pursue this course for the first few years, and—as the circular observes—to get no better results for their pains. The so-called "thorough" method defeats its own ends; the pupil is not in a position thoroughly to grasp the elements of any subject—Geometry or another—until he has advanced to a stage beyond those elements. Has anyone full understanding up to the very limit of his knowledge? Can anyone teach up to the limit of his knowledge? If a thorough grasp of elements were to be acquired by the study of elements alone, why should we wish for specialists to teach elementary mathematics in our schools? And the same holds good of school mathematics; if we spend three years over say the substance of two books of Euclid instead of six books, the pupil will understand those first two books not better, but less well, whatever his power of parrot-like repetition may be.

The circular suggests that definitions, with the exception of those which are matters of convention, are a luxury to be included or omitted at discretion. Now I do not care in the least about "the memorising of definitions intelligently arrived at and expressed in elegant form," but I certainly think it is advisable occasionally to ask pupils to explain, in their own words, terms which they already know how to use correctly—that of course must come first. It is no doubt true that progress in Geometry does not depend upon this, but it may be a distinct help towards progress in English, and we do not want to go back to the days when every subject was carefully fenced off from every other, as if the pupil's mind were similarly divided into little compartments. Even from a purely mathematical point of view, if we wish to help pupils to think clearly, it is well not to neglect opportunities for helping them to express clearly facts which they do know. It is certainly not a thing which comes easily to the English schoolgirl, and I have not been led to suppose that the schoolboy is more articulate than his sister.

There is one sentence in the circular to which I should like to call attention. "As the work advances it is more important to know the propositions themselves than to know their proofs." Most certainly, but let us beware that the pupils do not only get hold of a certain enunciation of a geometrical fact—often only one of many ways of stating it—instead of the fact itself. I wish examiners would occasionally vary the wording of the questions they set. I once set one as follows: "A and B are two points on a circle, and P and Q two other points on the same circle; prove that AB subtends either equal or supplementary angles at P and Q, distinguishing between the cases." Half the class

decided it was a complicated rider quite beyond them; most of the remainder gave no proof, but contented themselves with reasserting the same facts in the words to which, I suppose, they had been accustomed. Again, I should like to see more general questions set, so as to test a pupil's power of summing up portions of his knowledge, or of reviewing it as a whole and picking out what is relevant. For instance one might ask, "What do you know about the diagonals of a parallelogram? what properties have they in addition in certain special cases of a parallelogram?" or "With what data can a quadrilateral be proved to be a parallelogram?" or "Circles with a given radius are to be drawn touching a given straight line and a given circle; state how many there will be under the most favourable conditions, giving reasons for your answer."

On one point I distinctly disagree with the circular. The constant use of the word direction is urged upon us, and we are told that by means of the leading questions, "Are all vertical lines in the same direction?" "Are all horizontal lines?" we shall lead up to a clear conception of parallel lines as lines in the same direction. In my opinion, and according to my experience, we shall do nothing of the sort, and if we could, it would be a perfectly empty conception, as we can deduce nothing from it, without the further fact that the lines do not meet, which might as well have been taken as the definition in the first instance. If we follow the circular and do not require proofs of such fundamental facts as that corresponding angles are equal when a straight line cuts two parallel straight lines, I suppose we need not concern ourselves with the objections brought by mathematicians against proofs based upon the idea of direction, but it does seem to me that the circular ignores them in a rather lofty manner. Still my objections are more practical, and I speak from experience, as I got the idea from a German book more than seven years ago, and experimented with it before I had seen any of the new English text-books. It is distinctly a new idea to the child, and if he is to be led to it, he must be forcibly led. Every child I have ever questioned about parallel lines, has told me that they were always the same distance apart, and not one of them has had any difficulty in seeing that they therefore cannot meet; and it seems to me that this is quite sufficient to go upon. If the child has already a working idea, let us adopt that, and not lead him away—and up a blind alley—to another.

Mr. G. St. L. Carson said that a question of even more importance than the one under discussion was the more general issue whether geometry had any real claim to be a subject of general education. He was himself convinced that it had an overwhelming claim; but in these days of educational unrest it behoved mathematicians—and other teachers of single subjects—to answer such questions

to themselves and others. If the questions were answered in the negative, the only straightforward course would be to urge the addition of universal teaching of geometry; but if affirmatively, the reasons dictating the answer must provide the basis for any proper scheme. He imagined that the universal assent was based on the unique training in a combination of intuition, reasoning and expression which is found in the study of geometry. This being so, the present position should be reviewed in this light and no other.

He felt bound to say—speaking with full responsibility as one who had had a somewhat varied and extensive experience—that the whole value of the present study of geometry was small and decreasing if judged on this basis, and confirmation of this opinion had been afforded by an enquiry among the Professors and Examiners in the modern Universities, which showed an overwhelming majority against both the older and newer method as judged by their results in the later years of study. The only possible course in such circumstances was to ascertain, if possible, the causes of the failure; and he proposed finally to give his personal views on this matter very briefly.

In the first place he considered that Euclid had not been abolished as was generally imagined, but that his sway was almost as great as ever. His reason for saying this was that much time was still devoted to proofs of statements which were in themselves as obvious as the assumptions on which these proofs were based (for example, the propositions concerning the middle points of chords of circles are obvious to anyone who has the least sense of geometry), so that an endeavour is still being made to reduce the number of intuitions below that natural to a normal mind. The process of analysing such intuitions so as to reduce them to a minimum is the Euclidean method; it might have its charms for adults, but was utterly unsuitable for children, who are soon confused and wearied by long arguments which end in familiar conclusions. Another striking example was the persistence of the angle propositions on parallels (Euclid I. 27, 28, 29). These are truisms to anyone who appreciates the terms "direction," "parallel"; if a child should not agree with them, he must be made to understand these terms, when agreement must follow.

His second criticism was that instrumental drawing—so valuable in itself—was being misused to such an extent that it might be said that a tyranny of instruments had been superimposed upon the Euclidean tyranny to which he had just referred Take for example questions of the type, "Draw a pair of parallel lines and measure the alternate angles. Repeat several times and tabulate your results. What do you observe?" The natural intuitions from rough general experience and the sense of symmetry to

which he had referred were here being stifled by misleading and abnormal processes which led to mental empiricism of the most vicious description. In his own teaching he used instruments incessantly as a check upon and illustration of pure reasoning and for the performance of numerical constructions, but he desired to say with some emphasis that in his opinion all instruments—even rulers—were a serious hindrance in arriving at those intuitions which must be the basis of the subject. The matter might be put most tersely by saying that such bases should—for any subject—be entirely qualitative and not quantitative if this were possible, and in geometry it was possible.

Finally, while not wishing to cavil unduly, he felt bound to refer to the entire neglect of the curves other than the circle, easy and illuminating as they are in many respects, and also to the failure to introduce three dimensional ideas throughout the school course. The value of these ideas was universally admitted, and the fact that it had remained as a pious expression of opinion was, he imagined, one more indication of the persistence of Euclidean ideas.

Dr. T. P. Nunn: I am cordially in agreement with Mr. Siddons and my predecessors in the discussion in welcoming the circular as a very valuable document. It does not of course say the final word upon the pedagogy of geometry, but it is an interesting and stimulating contribution to what we must believe is still a relatively early stage of our inquiries into this subject.

Before we can reach really satisfactory views on the teaching of geometry, we must adopt some definite conclusions on a difficult and important question with which the circular does not directly deal—namely, the intimate nature of the differences between the reasoning activity of boys and girls at different stages of their development. To this question it would be impossible to make a useful contribution in a few minutes; I pass therefore to smaller topics. I agree with Mr. Siddons in commending the plan by which a certain group of propositions is placed outside the deductive scheme. It would be an excellent thing if this plan were adopted in school examinations as well as in those conducted by public bodies. I have known cases in which an examinee who used the congruence proposition (familiar to us as Euclid I. 8) in order to prove the equality of the base angles of an isosceles triangle lost all credit for his answer because in the class text-book the latter proposition precedes the former. This treatment of the subject tends to be altogether destructive of a boy's respect for geometrical reasoning as a process of investigation of space. I do not, however, think that it is necessary to ask boys to accept the proposition dealing with parallels and congruence on the authority of better thinkers than themselves. On the contrary, the difficulty with these

propositions is that they are to the boy of twelve such obvious generalisations of his everyday spatial experiences that he can hardly understand what you mean when you ask him to prove them. Thus, in the Demonstration School of the London Day Training College, where no text-book prejudices the relations between teacher and pupil, my colleagues and I, who have used practical problems in heights and distances as the motive for the construction of triangles to given data, have found surprising difficulty in persuading the boys that the validity of their constructions is a matter worth calling into question. Even if their drawings give different results for the distance across the street, they do not question the adequacy of the method employed, only the accuracy of their drawings.

There is another set of implicit geometrical generalisations which I think should be added to the circular's group of preliminary propositions; I allude to the simpler properties of similar triangles. It is clear that, implicitly, every boy believes confidently in these principles: they are involved in the drawing of every diagram to scale, and in the use of every map. On the other hand, they differ from the congruence propositions, in that they form a much better starting point for an attempt to teach a boy to justify his geometrical convictions by analysis. Thus, if the members of a class represent on different scales the data of an attempt to measure the height of a tree, there is an adequate motive for inquiring how it comes about that the various representations all lead to the same result. In the course of this investigation—which I think boys invariably find interesting—the principle of congruence of triangles by super-position would enter naturally.

I agree with Mr. Siddons in regretting that the circular should discourage riders in the earlier stage. Much depends here upon the way in which the rider is presented to the boy. During this last week I have been watching classes whose syllabus in mathematics contains a large element of practical work. In the course of their elementary studies in geometry the boys are given small mirrors, pins and drawing paper, and are asked to find out "How a looking-glass works." They first establish the fact that the image of the pin is just as far behind the mirror as the pin itself is in front. Next they find the equality of the angles of incidence and reflection. They are then asked, "How could you have foreseen the second result knowing the former?"; "How long must a mirror be if you are to see the whole of your height in it?" etc. Questions like this, leading to results to be verified by experiment, are very much more stimulating to the boy than abstract geometrical queries expressed in terms of a conventional *ABC* and *PQR*.

Mr. J. J. Garstang (Bedales School): Many attempts have been

made to base the treatment of parallels in geometry on the notion of direction. History does not supply evidence that any such attempt has been successful. Without attempting here to give a complete account of this historical evidence, we may put forward the following instances.

In 1868, Mr. J. M. Wilson published a new book on geometry containing this method of direction. This book was reviewed in the *Athenæum* for July 18, 1868, by the late Prof. De Morgan. We quote an extract: "There is a covert notion of direction, which, though only defined with reference to lines which meet, is straightway transferred to lines which do not. According to the definition, direction is a relation of lines which *do* meet, and yet lines which have the same direction can be lines which *never* meet. . . . How do you know, we ask, that lines which have the same direction never meet? Answer—Lines which meet have *different* directions. We know they have; but how do we know that, under the definition given, the relation called *direction* has any application at all to lines which never meet? The notion of limits may give an answer; but what is a system of geometry which introduces continuity and limits to the mind as yet untaught to think of space and of magnitude?" * Mr. Wilson withdrew his method before 1878.

In Jan. 1891, at the annual meeting of the A.I.G.T.—the parent Society of the present Mathematical Association—there was a long discussion on the proposal of Mr. E. T. Dixon to use this notion of direction for parallels. The President and the meeting were quite adverse to the suggestions.

On p. 4, line 2, Circular 711 states that by some preliminary work with vertical and horizontal lines, it is possible to lead "to a clear conception of parallels as lines in the same direction." The circular does not answer De Morgan's question, nor refute his criticism, nor make any reference to any published paper or book where such refutation can be found. In absence of this refutation, the statement of Circular 711 cannot be accepted as correct; and the conclusion of the greatest experts on this question still stands, namely, that the introduction of the notion of direction as the basis for parallels is fundamentally fallacious and ought to be banished from school teaching.

Mr. J. G. Hamilton: The first thing that must strike anyone going straight from the world of life into that of school is the unreality of so much of the work of the class room, and the appalling waste of misdirected energy in Mathematical teaching. This is inevitable so long as it is dominated by the academic spirit, and, as this geometry circular amply proves, it is not just a question of modern *versus* ancient geometry; it is largely one

* Quoted by Prof. Cajori in the *Teaching and History of Mathematics in the United States*, p. 382.

of the mental attitude and outlook of the teacher. One is led to wonder whether the undoubted loss due to the narrow limitation of geometry to the technics of a mere school subject has been at all adequately realised, any serious attempt made to estimate the gain that would at once arise from making it emerge as far as possible in response to the demands of the ever widening experience of boy and girl. Geometry, pre-eminently, has always been imposed on the pupil from without, and perhaps one of the greatest merits of this official document is its insistence on the standpoint being that of the pupil and not that of the professor. With a breeziness particularly refreshing it sets forth the absolute necessity of bringing the school treatment of geometry into close relation with the natural activities and living interests of the pupil, and of leading him to an intelligent conception of the geometrical significance of his surroundings.

This vitalising of the work is impossible under the traditional system with its book work, propositions and the like, and its lack of appeal to the valuable experience, the wonder and imagination of a boy or girl. Whewell's attempt at imposing this soul-destroying proposition system on the teaching of mechanics is a thing of the past, and no one now would ever dream of reverting to it. So geometry must develop. It will be more and more extensively based on experimental work, carefully devised and purposeful, and not so narrowly limited as at present either in conception or in outlook. By graph work, by mechanism, illustrative of continuous change of form, will a learner be freed from the trammels and unsuggestiveness of the static figure and the cramping limitations of plane geometry. Opposition to this development and disinclination to adopt its methods can only come from those who do not know its possibilities, or who are under the too common misapprehension with respect to it. This misconception is that it is simply empiricism, and that it makes no demand on the reasoning powers or the imagination. In point of fact, it is just when you leave pure theory and come to its practical applications that the real difficulty begins. It is while you are in the academic stage that everything is easy; when you go farther afield you find yourself compelled, not only to justify the fundamental principles, but also to consider what modifications may be necessary in practice. Not only is this scientific method much more fruitful than the traditional in securing the best kind of results, but it is far more provocative of original work, which, as the circular strongly insists on, is the one thing of value worth making for. Moreover, the pupil enjoys his work, because he feels something of the pleasure of the creative artist. The circular falls short of what it might have done, in not giving a stronger impression that the experimental and experiential method is also of very great value

in later stages as well as in earlier, for here again it gives a greater challenge to thought and is infinitely more suggestive than the method it displaces, while throughout it supplies a "background of wide experience," which every teacher who knows his business knows the value of.

In conclusion, I contend that the circular is quite sound in its attitude towards the relation of theory and practice, while the Mathematical Association is entirely in the wrong in even suggesting by its device of parallel courses that practice and theory may be divorced. The evils of separation have been abundantly demonstrated in the past, and the effects on certain examinations have proved most unfortunate. From the very first, practical work need never be dissevered at all from reasoning, and it is to be remembered that it is by the happy combination of practice and theory, and the continual interaction between the two that discoveries are made, and boys and girls get the pleasure and profit in their way just as great discoverers do.

With the time at my disposal I have only been able to give in briefest thumb-nail sketch one or two of the ideas by which I have been swayed in my own work. I must refer those who are doubtful of my meaning to an article of mine in the *Mathematical Gazette* for March 1907, which gives more fully, and supplements the main positions of my present criticism, and much of which is applicable in this discussion.

Mr. F. J. W. Whipple attached great importance to continual progress in learning Geometry. To go back to the beginning to learn the proofs of facts with which he had become familiar in a preliminary course was bound to damp a boy's enthusiasm. On the other hand, the course ought to show that the subject was a connected science and not a collection of isolated facts.

NOTICE.

Members of the Association who desire to be registered as members of the London Branch are asked to send an intimation of their wish to P. Abbott, 5 West View, Highgate Hill, N. (provided they have not already done so). They will then receive all notices, etc., of the Branch.

NORTH WALES BRANCH.

A MEETING of the North Wales Branch was held on June 18th at Llanberis. The discussion on examinations which had been begun at the previous meeting was reopened by Miss Prior, Girls' County School, Bangor, who dealt in turn with the points raised in Prof. Bryan's paper, from the point of view of the school-girl.

1. There should be no choice of questions, at any rate up to matriculation, if the syllabus is definite, but choice is necessary in the case of a wide syllabus, as in history, literature and languages. A child under twelve has enough to do to answer a paper of questions without choice.

2. Too much memorising, which is not the object of mathematics. Geometrical definitions and certain propositions ought to be tested, and in higher work proofs of formulae given and then the formulae left to the memory. Papers should contain book work, but not so much as to enable a candidate to pass on that part alone.

3. In marking papers neatness and accuracy to be considered, for untidy workers are generally loose thinkers. A good method is to mark a paper on a maximum of 90, adding an extra 5 if no careless mistakes are made, and another extra 5 for brilliancy in working. Marks may be deducted from these extra ones for untidiness or other reasons.

4. The importance of uniformity in examinations, especially when examiners are changed; a child ought not to be confronted with an entirely different paper.

In the subsequent discussion Prof. Bryan considered that originality is pressed too much in making up examination papers, and urged the setting of book work with slight modifications, also the importance of giving easy practical applications of geometrical propositions. Miss Griffiths mentioned the value of writing out propositions, and said that girls are more frightened than boys in examinations and are discouraged by difficult questions. Mr. Foster, Llanberis County School, thought that not much choice is necessary in pass papers and that arithmetic questions should be straightforward and not all problems. Mr. Madoc Jones considered that no choice of questions binds a teacher down unnecessarily, and said that most girls memorise too much, always looking for a model by which to answer a question, but that some girls are very good at mathematics.

SOUTHAMPTON AND DISTRICT MATHEMATICAL SOCIETY.

A MEETING was held at King Edward VI. Grammar School on Friday evening, March 11th. The president, Professor E. L. Watkin took the chair. The meeting was considered as the First Annual General Meeting of the new Society. Professor Watkin was re-elected president for 1910 and C. H. Holmes hon. secretary and treasurer.

A paper on logarithms was read by Dr. Fenwick, headmaster of Bournemouth School, which took the form of a lesson on logarithms to beginners, similar to those lessons which Dr. Fenwick had proved to be successful at his own school. The index laws were first proved, and then any number was shown to be capable of representation as 10 to a certain power equal to the log of the number. All operations on numbers could then be conducted by operating with their substitutes. Thus pupils could be working with logarithms without knowing any technical terms such as characteristic and mantissa.

In the discussion which followed Prof. Watkin strongly advocated the use of a graph, and in practical work with logs strongly deprecated all useless labour, such as writing logs in lines instead of in columns.

Mr. Holmes suggested that all subtractions of logs should be done one logarithm at a time. He also showed a simple method of constructing a table of logs of numbers from 1 to 10 by elementary arithmetic without extracting square roots.

Mr. Thomas was strongly in favour of a graph. He showed a graph prepared by himself which gave three figure logs of the numbers from 1 to 10. The pts. on the graph were obtained by Perry and Edser's method.

A MEETING was held at the Hartley University College on Friday evening, May 13th, the president, Professor E. L. Watkin, being in the chair. Twelve members attended.

A discussion on the Board of Education circular No. 711 was initiated by C. Godfrey, Esq., M.A., of the Royal Naval College. The chief points of Mr. Godfrey's address were as follows:

The reform policy was approved—the drawing of geometrical patterns was advocated. It is better to go slow than to go fast as advocated in the circular. Intuition should be encouraged, even when the work is entirely theoretical. The treatment of the fundamental propositions as advocated in the circular was approved. The sequence trouble is not serious. It is to be regretted that the Board did not give a stronger lead with regard to solid geometry. In France, Austria, and Germany two and three dimensional geometry go hand in hand.

In the discussion which followed the members generally were in agreement with Mr. Godfrey's position. Mr. Phillips, of the Hartley College Engineering classes, thought that the sequence difficulty was a real one, and Dr. Fenwick thought that it was better to go quickly through the geometrical course as advised in the circular.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

Sir,—I have read with much interest the important letter by Miss Edith Stoney in your last issue, in which she criticises the method for calculating the coefficient of linear expansion recommended by the Joint Committee of the Mathematical Association and the Association of Public School Science Masters.

Whenever we consider the various methods of teaching any subject, it is important to distinguish between those suited to the mature and those best adapted to the immature mind: with the former it is possible, and since it saves time, probably wiser; to reason from the abstract to the concrete, *i.e.* to adopt mathematical methods, and such pupils will not be confused by the exact treatment of problems like that mentioned: but with those whose minds are as yet immature the teacher is forced to choose between letting them reason from the concrete to the abstract or being content that they shall adopt rule of thumb methods.

In teaching linear expansion to the younger boys in our Public Schools (and the report, in urging that lessons in heat shall precede any in chemistry, is clearly contemplating such pupils) the method adopted is first to give a demonstration of the qualitative fact of expansion, and then make the pupils themselves measure the amount; when this has been done the results obtained in the laboratory are used to furnish problems in the class-room. The form of apparatus used for determining the coefficient of linear expansion varies much, but the principles of almost all depends upon the use of a micrometer screw, and consequently the limit of accuracy which can be obtained is 1/100 millimetre; under these conditions no difference can be detected between the results obtained when the initial temperature of the rod varies, and it would only confuse the pupil to be told he should consider this. A few weeks later the expansion of gases is considered, and here it is at once found from the results of experiments made by the pupils themselves that the initial temperature makes a very considerable difference in the values obtained: the teacher may, and in my opinion should, when explaining this, point out that the assumption tacitly made when considering the expansion of solids is only approximately true.

When the pupil comes back to the subject two or three years later his mind will have developed, and he will have no difficulty in understanding either the exact scientific facts or the mathematical treatment of them;

at this stage, I need hardly say, the method mentioned by Miss Stoney is the correct one to adopt.—Yours faithfully,

DOUGLAS P. BERRIDGE,
Hon. Sec. Association of Public School Science Masters.

Malvern.

TO THE EDITOR OF THE *Mathematical Gazette.*

SIR,—My letter was written under the influence of warm enthusiasm for the main text of the Report on the *closer correlation* of Mathematics and Physics teaching in our schools. On p. 2 of that Report the chief obstacle to such correlation is assumed to be the lack of laboratory training amongst our mathematical teachers. I fear I venture to think it is also to some degree due to unwillingness amongst our science teachers to spend the necessary time in class in order to make our children use practically the knowledge which the mathematical teachers have (probably) already given them theoretically. Surely it is a delight to most children to find that their dry mathematics "are of some use"! Boys, and most girls from good modern schools, at say 13 years of age, know enough algebra to be able to multiply $(1+x)(1+y)$ or to divide $\frac{1}{1+x}$. It is not proposed, I believe, that such a subject as Heat should be taught at a younger age than about thirteen.

This is very elementary compared with the use of logarithms or a slide rule, and yet it is all the mathematics which I presuppose in my example and which is considered too abstract by Mr. Berridge.

Assuming that this amount of abstract knowledge has already been given, then my point was that the Physics teacher could well correlate it and shew its use in a concrete form. Certainly, to train the class in the practical use of their mathematical knowledge will be at the cost of a little time at first—though I believe at a great saving of time in the end. Soundness of training and not questions of time are however the chief consideration at such junior ages. This training will also shew the students how to allow for probable errors in their results due to micrometer screws, et cetera.

Of course it is true that experimental errors are likely to be greater in such experiments than those from using approximate formula; but are we to state so *ex cathedra* to our children, or are we to give them the means of estimating the effects of the various errors in the results for themselves? Using no such elementary mathematics as I suggested, I do not know how Mr. Berridge proposes to explain to a class that the vague and inaccurate theoretical method suggested in the example given in the Report is justified by probable errors in the micrometer screw! It is just those of my students who find Mathematics and Physics difficult who are most confused and made to feel unsound if I tell them all their work a few weeks before depended on tacit assumptions which I had induced them to slither through unconsciously.

If the school teaching of Mathematics and Physics is to be correlated, why not begin from the earliest stages of the Physics and use the Mathematics the children already know? The chief object of the Report may be to get our Mathematical teachers to make use of concrete Physical illustrations, but is not also one object of the Report to encourage us to continue no longer to teach the "emasculated kind of Physics without Mathematics which would not give a headache to a caterpillar" which Sir J. J. Thomson warned us is the present tendency?

EDITH A. STONEY,
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REVIEWS.

A Course of Pure Mathematics. By G. H. HARDY, M.A. 12s. net. 1908. (Cambridge: University Press.)

The title of Mr. Hardy's book is a little perplexing. Anyone who was unfamiliar with the author's other work might expect it to be a compendium of pure mathematics, or of such comparatively elementary parts of the subject as could be compressed into a single volume of reasonable size. Thus it might consist of enunciations and proofs in outline of propositions in elementary plane geometry, geometrical and analytical conics, algebra, trigonometry, and differential and integral calculus. Any such expectation would be at once falsified by an inspection of the book. It is, in fact, substantially an elementary treatise on what continental writers call analysis, a body of doctrine of which the central idea is the *limit*. It is remarkable that, as far as I know, there is no English book on the subject, though its contents are dealt with, more or less imperfectly, in the later chapters of books on algebra and trigonometry and in treatises on the differential and integral calculus. Mr. Hardy's book is much more elementary in scope than the familiar treatises or courses on analysis with which Goursat, Jordan, Picard, Humbert and other eminent French mathematicians have enriched our literature. Tannery's *Leçons d'Algèbre et d'Analyse* is perhaps the book nearest to Mr. Hardy's in scope, though it is wholly dissimilar in many respects.

The mere transference to a single book of matter which is usually found in three or four has the advantage of bringing out points of connection which are otherwise liable to be obscured, and from the point of view of the student avoids a good deal of repetition. A beginner, who has read about the exponential theorem in Chrystal's *Algebra*, may be a little distressed when he is expected to read it all over again in a different form in Hobson's *Trigonometry*, and again when at a later stage he meets a more or less satisfactory treatment in a differential calculus. Moreover, the writer who is mainly interested in obtaining trigonometrical or algebraic formulae is very liable to slur over some of the delicate logical points that are really involved in quite elementary propositions about series and products. Mr. Hardy is primarily interested in the fundamental ideas underlying infinitesimal analysis, so that he is naturally led to treat them fully and carefully, using the trigonometrical and other applications as illustrations rather than regarding them as ends in themselves.

It may be convenient at this point to give some idea in rather more detail of the ground covered.

A short preliminary chapter explains the notion of real rational and irrational numbers, the latter not being based on strict arithmetical ideas, but involving an appeal to geometrical intuition. Then follows a discussion on the meaning of a function, and its graphical representation by a curve, involving a treatment of the elements of analytical geometry; a large number of examples are given which illustrate the idea on which the author rightly lays much stress, that a function may not be defined at certain points or for certain ranges of the variable. The next chapter defines complex numbers, practically as vectors, establishes rules for their manipulation, and leads up to Demoivre's theorem and a number of the familiar trigonometrical consequences. The two important chapters which follow introduce and illustrate the idea of a limit, first for the case of a positive integer tending to infinity, and then for the case of a continuous real variable. The earlier of these chapters includes a first discussion of convergent and divergent series, and the second contains proofs of two fundamental propositions frequently omitted as "obvious," viz., that a continuous function passes through all values intermediate to any two of its values, and that a continuous function attains its upper and lower limits. The two following chapters deal with differential and integral calculus proper, and includes, in addition to first principles, the differentiation and integration of the simpler algebraic and transcendental functions, maxima and minima, Taylor's theorem and some applications to areas and arcs. The convergence of infinite series and integrals is dealt with in a single chapter. Some half-dozen of the more fundamental tests of convergency are given, including the very useful integral-calculus test of Maclaurin and Cauchy,

which has been much neglected by English writers. Several of the more important properties of absolute convergence are established, and there is a slight treatment of complex power series; but infinite products, double series and uniform convergence are omitted.

The two final chapters, placed, it may be observed, after the chapters on differentiation and integration, are devoted to a detailed exposition of the theory of the logarithm and exponential function, first for real and afterwards for complex arguments. Mr. Hardy defines $\log x$ as the integral $\int_1^x \frac{dt}{t}$, and e^x as the inverse function; and then deduces the familiar series and limits. The more usual procedure of starting with e^x defined as a series is briefly sketched as an alternative method. Though Mr. Hardy's procedure has been suggested before, I do not remember to have seen it worked out at length, and it constitutes one of the most interesting novelties in his book. One great advantage is that the rearrangement of a double-series involved in the usual proof of the series for $\log(1+x)$, and the further appeal to Abel's continuity theorem which is usually required to establish the series for the case $x=1$, are avoided; on the other hand, Mr. Hardy's treatment of complex arguments involves the integration of a complex variable along a curvilinear path, which seem to me to belong really to a much more advanced order of ideas. There is the further disadvantage that Mr. Hardy's definitions have no apparent connection with the elementary idea of the logarithm, which every reader of his book will have met in connection with numerical calculation, and though the gap is soon bridged over, the abrupt change of definition is likely to be puzzling and disconcerting to a beginner. I am not, therefore, at present convinced of the superiority of Mr. Hardy's procedure, but I should like more experience of teaching it before expressing a decided opinion.

Two appendices contain a proof that every equation has a root, and some discussion of the meaning and difficulty of double-limit problems.

There is naturally much room for difference of opinion as to the proper content of a book of this sort, where there is no tradition as a guide, and probably any teacher would wish certain things added and others omitted. For my own part, though I am in substantial agreement with Mr. Hardy's scheme, I could gladly spare the few pages on elementary analytical conics, which would surely be familiar to any one who is capable of reading the book at all, and I should like a rather fuller treatment of the integration of algebraic functions. Mr. Hardy goes so far in his treatment of the integration of rational functions and functions involving a square root of a quadratic, that it seems a pity that he did not add a little more and so complete the subject, as far as is needed for ordinary purposes.

I have suggested at the beginning of the review that the chief feature of Mr. Hardy's book was a rearrangement into a single book of matter usually treated in three or four. Far more important is the fact that—to put it shortly—when Mr. Hardy sets out to prove something, then, unlike the writers of too many widely read text books, he really does prove it. Probably a really acute critic reading the book would discover here and there gaps in the logic, but I feel sure that these would prove to be few and trivial; and if the book is widely read, I for one shall hope to avoid in the future the many weary hours that have usually to be spent in convincing University students that "proofs" which they have laboriously learned at school are little better than nonsense.

Apart from merely trivial slips and misprints (chiefly in the examples) I have only noticed one substantial lapse. In giving Cauchy's geometrical proof that every equation has a root, Mr. Hardy is at pains to avoid the traditional but subtle error which involves the idea of uniform continuity, but makes the equally traditional but much simpler mistake of omitting to postulate the continuity of the amplitude, though at one point he rather oddly "proves" it. The essence of Cauchy's proof is to show that under certain conditions the amplitude of a function does or does not increase by a multiple of 2π. But the amplitude is by definition indeterminate to the extent of an arbitrary additive multiple of 2π. Surely it is as meaningless to speak of such a function being increased by 2π, as of a change of sign of a number which is merely defined as $\pm x$? The omission can, of course, be remedied in a couple of lines, but a hostile critic might plausibly maintain that the proof as printed is meaningless.

A few passages strike me as rather obscure. The discussion (p. 279) of the

ambiguities in integrals involving $\tan^{-1}x$ would leave the student helpless in dealing with such a familiar integral as $\int_0^{2\pi} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$; would not an appeal to continuity be much more satisfactory? And the proof of the fundamental theorem of the integral calculus, though contained in §§ 137-142, is nowhere very clearly enunciated or proved. It would also be better to explain the unfamiliar notations *arc sin x*, etc., in the text as well as in the preface; for students commonly begin their reading of a book where many reviewers end it, *after* the preface.

An objection which may strike many readers is the size of the volume (xv + 428 pp.). This seems to me more apparent than real, as the book is clearly to be read rather than "got up." Many sections, it is true, are "bookwork" in the conventional sense, but by far the larger part is explanatory matter aiming at the elucidation from different points of view of fundamental ideas, and here there is no real objection to some diffuseness. The great Poincaré describes himself in one of his papers as *un peu bavard*; many of us would have been glad if the same epithet could have been applied to some of his other writings. Similarly, the number of examples may appear excessive; a student who expected to work through them systematically might indeed shrink from the prospect; the intelligent teacher will, of course, select. An excellent feature is the great variety of analytically very simple examples which illustrate what are generally regarded as abnormalities of functions. They effectually dispel the illusion that functions which are finite but discontinuous, or continuous but not differentiable, etc., are only to be represented by elaborate series or definite integrals. There is something to be learnt from so simple a function as x/x, and an analytical formula (p. 151) for the number of days according to the Gregorian calendar in the year n A.D. is quite simple. ARTHUR BERRY.

Théorie des Marées. H. POINCARÉ. Tome III. of the "Leçons de Mécanique Céleste." 1910. (Gauthier-Villars.)

Everything that M. Poincaré writes is delightful, and the only fault to be found with the present volume of 469 pages is that there is no index, and only a very meagre list of contents. But after all a romance needs no index, and there is much of the charm of romance in these pages. Even the mathematical formulae have by some magical process lost their long tails of crabbed numbers (and my goodness! what numerical monstrosities the tides can produce!), and dance and transform themselves like fairy things at the hands of the wizard writer. And yet the serious business is not neglected. We are left with a tolerably clear view of the present practical possibilities, which if not very hopeful is at any rate not hopeless. As regards the most practical problem of all, that of accounting numerically for the ocean tides, M. Poincaré seems to think favourably of the suggestions of Mr. R. A. Harris of the U.S. Coast Survey, which are somewhat as follows. Suppose we had a number of oceans in closed basins of regular form on the earth: they would have free periods of vibration. If the depth of a basin were such that one of these periods nearly coincided with a lunar (or solar) period, the corresponding oscillations would be large. If the basins were not closed, but all connected, these particularly large oscillations would spread through neighbouring basins, and probably dominate the whole system of movements. Mr. Harris has accordingly looked for localities where the depth of the ocean bears a suitable ratio to its surface dimensions. To supplement the timid guidance of theory in such matters he has had recourse to experiments in artificial basins, and has therefrom formulated certain general considerations. M. Poincaré, as already remarked, thinks favourably of the central idea, but does not trust these deductions from experiment (one of them he shews to be wrong). He thinks that it would be possible to get closer to the facts by actual integration over the critical areas. He admits that the labour would be enormous, but he thinks it can be faced, and that a suitable method is furnished by the discoveries of Fredholm. The tenth chapter (of 70 pages) in which he expounds this method, is harder reading than the rest of the work, especially to those whose work has not lain recently in such directions; but it carries the conviction that it is at any rate easier reading than the original: and one looks at the pages with great respect when one comes to Poincaré's remark that it is this method or none ("Il n'en existe pas d'autre," p. 296).

It is gratifying to Cambridge men to find Hough's work given a high place. Poincaré quotes several times his result that friction plays such a small part in ocean tides, requiring 10 to 20 years to reduce them in the ratio 1 to e: and he early draws attention to the fundamental distinction, due to Hough, between two kinds of statical tides (one in which there is a simple deformation with no currents, and the other in which currents exist). One result of the smallness of the friction is that it is 100,000 times too small to explain any secular acceleration of the moon, which must therefore be due entirely to bodily tides in the earth, as Darwin has indicated. Another point in connection with secular acceleration is worth noting. An elementary explanation often given is something like this:

(*a*) if there were no friction, it would be high water under the moon;

(*b*) owing to friction the high water lags behind the moon;

(*c*) there is thus a couple due to the attraction of the moon on the two high waters.

Now, firstly, the lag is not due to friction: that correction perhaps need not have been mentioned; but secondly, though the lag exists and is due to other causes, "there ain't no sich" couple. M. Poincaré proves in a few lines of his elegant analysis that the couple is zero (pp. 451, 452). Among many instances of neat work that on pages 103-5 may be quoted, where it is shewn that by using a conformal representation of the sphere on a map a simple form of the equation of continuity can be obtained, much simpler than for the sphere itself. H. H. T.

Interpolationsrechnung. By T. N. Thiele. Pp. xii+175. 10 m. 1909. (Teubner.)

The idea of the author of this treatise is that interpolation is a practical subject which might well be used to round off a course of elementary mathematics and at the same time introduce ideas which are of fundamental importance in the infinitesimal calculus and in the theory of infinite series. He says that interpolation occupies a central position on the arithmetical side analogous to that of trigonometry on the geometrical side of mathematics. Examination of his book seems to us to show that however attractive the notion may be *a priori* it is impossible to carry it out in practice. The reason is that there is no rule of interpolation which is exactly applicable to every function. Formulae may be suggested by interpolation, the exponential theorem, the general binomial theorem, and Taylor's theorem, for instance, but in each case an independent proof is necessary. The mathematician who is familiar with those results may find it interesting to correlate them with the theory of interpolation, but the reverse process seems to us foredoomed to failure. Taken on its own merits, without regard to the preface, the book will be found to offer a great variety of methods to the computer who has sufficient mathematical equipment to decide on the most suitable one for use in any particular case. F. J.

A Course of Plane Geometry for Advanced Students. Part II. By C. V. Durell. Pp. xiv+357. 7s. 6d. net. 1910. (Macmillan.)

Mr. Durell's second volume is a worthy successor to the first, and shows that the author possesses a thorough grasp of his subject and the power of presenting it in an eminently attractive manner. At the same time there are some signs of haste, and in some cases a too brief or too scattered treatment of many important theorems, and a beginner will find difficulty in grasping the full meaning of some of them, and in providing proofs where proofs have been "left to the reader." This happens in 84 of the 225 theorems of the book, *i.e.* more than a third of the whole number. But students who already have some acquaintance with the subject, or who have a teacher possessing the knowledge requisite to fill the gaps, will find the book full of interest and fruitful suggestion.

The author first introduces the reader to various properties of imaginary points and lines, the circular points at infinity and isotropic lines in particular, explaining their importance in connection with homography, and shows how to employ analytical ideas generally in connection with geometry, and par-

ticularly in connection with homographic transformation, giving plenty of food for thought in such theorems as *e.g.* that "concentric circles are transformed into conics having double contact."

Then comes a chapter on orthogonal projection and derived properties of conics, followed by an important synopsis with illustrations (pp. 51-64), written by Mr. A. E. Broomfield, of various methods employed in practical solid geometry. This is followed by chapters on conical and general projection and their application to the conic with its fundamental cross-ratio and harmonic properties, excellently given for the most part, but with some important theorems only sketched or somewhat scattered and left to the reader to complete or collect. This is specially the case in connection with the converses of the dual theorems of p. 96 (No. 57), which establish the cross-ratio properties of conic pencils and ranges on tangents. These converses are alluded to on p. 115, but are not completely proved geometrically till pp. 251, 252. A neat analytical proof of one converse is given in Ex. 149, p. 119, and by using line-coordinates as in Ex. 235, p. 201, it becomes a proof of the dual converse. The other proof, pp. 116, 117, assumes that a conic can be drawn through 5 points or touching 5 lines. There is an analytical proof of the first of these lemmas on p. 79, but a geometrical proof is not found till p. 259, the dual lemma following on the next page, the proof of it, however, being left to the reader. It seems a pity that all these theorems are not given in Chapter IV., to which they belong. They do not appear to present difficulties great enough to necessitate their postponement.

A chapter is devoted to Pascal's, Brianchon's, and Carnot's theorems, the so-called Newton's theorem (which, however, is certainly as old as Apollonius), and others of importance. The proofs of the two last mentioned seem needlessly long and difficult. They can be proved in a few lines by bringing in parallel semi-diameters and using properties easily established by orthogonal projection, whereas the author requires nearly a page for Carnot's, and derives Newton's (that the ratio of products of segments of pairs of intersecting chords fixed in direction is constant) from Carnot.

The next chapter, on Reciprocation, forms a valuable summary for a student revising his knowledge of this method, and concludes with an analytical treatment by point and line coordinates.

The analytical treatment of homographic ranges and pencils is well given in Chapter VIII., followed by a careful geometrical exposition, with simple methods of finding corresponding points and double points in the case of cobasal ranges, and with examples, worked out, of their use in solving problems. It seems strange that the method of solving quadratic problems by finding and using double points should be called (p. 232) construction by "trial and error," for surely it is anything but that. It is an old term, but it seems quite inappropriate to this accurate and elegant method. By the way, it is desirable that students should complete the required constructions, as a rule, and not stop short at indicating the double points on which the solution depends. The author well exemplifies both complete and incomplete solutions (Exs. I. and II., pp. 233-235). In this chapter, p. 208, we notice the statement that the existence of the equation $pxx'+qx+rx'+s=0$ is the necessary and sufficient condition that the ranges given by x and x' shall be homographic. This is true, if you can get two ranges from it, but it fails if the expression is factorizable, *i.e.* if $ps=qr$.

Chapter IX. is devoted to homographic properties of conics. Chapters X., XI. deal with involution and involution properties of conics, and the last chapter (XII.) gives some important miscellaneous theorems. And there is a good index at the end, besides a careful table of contents at the beginning.

In most of the chapters there are valuable historical notes, which add greatly to the interest and solidarity of the subject, and there is a magnificent collection of over 1800 examples, as to the most advantageous use of which the author makes a valuable suggestion to the teacher in the preface.

The book is a notable and most attractive addition to the literature of the subject, and we wish it all success. A. L.

Elementary Solid Geometry. Part III. Polyhedrons. By T. SUNDARA ROW. Trichinopoly: St. Joseph's College Press. 1909. Pp. 91.

The author's book on "Paper Folding" is widely known. The present work contains four sections on polyhedrons in general, semi regular polyhedrons, star polyhedrons, and the method of crystallography (viz., the index system). The numerical details and accurate diagram of numerous solids would be very helpful in constructing models. The Oxford Dictionary decides a doubt as to the plural of polyhedron in favour of both forms. C. S. J.

Geometry. Part III. By W. P. WORKMAN and A. G. CRACKNELL. Pp. 72. Clive & Co. 1908.

This part of the complete work contains the substance of Euc. XI. The book begins with a few notes on perspective as an aid to the representation of solid figures. The question whether "plan and elevation" or "isometric projection" would not for beginners be both easier and more definitely geometrical will no doubt be answered by experience when the geometrical reformers succeed in obtaining for solid geometry its proper place in elementary instruction. Good yet simple diagrams, and the scholarly treatment which everyone expects from the authors, characterise this little book.
C. S. J.

Practical Solid Geometry. By the REV. P. W. UNWIN, Cheltenham College. London: G. Bell & Sons. 1909. Pp. xii+267. IV. Price 4s. 6d.

When our English practice differs from that of the civilised world we may be right, but in mathematics at any rate the probability when judged by the past seems against us. Descriptive geometry furnishes a characteristic instance of an insular peculiarity. A glance at a catalogue of a single German firm shows 24 books on this subject. Italy, France, and America have each quite recently produced notable works on the same branch of mathematics. In this country the subject as a branch of mathematical education has for many years been practically extinct. Within the last few years eminent publishers in this country have declined even to discuss the question of a translation of one of the leading continental text-books: and inasmuch as publishers are not State endowed educational agencies, with a statutory exemption from the operation of the Bankruptcy Acts, their decision was in all likelihood correct. The appearance of a book on the subject, originating in one of the most conservative of our public schools, and still more the promise of a sequel, are therefore sources of special gratification to one who agrees with the author that those who spend even a little time on the subject find that "the assistance derived from its study, in the other branches of their work, has well repaid the time devoted to it, for the study of the problems of descriptive geometry develops the power of grasping the important points of mathematical problems involving the consideration of figures which lie in different planes." That a first book on solid geometry should fuse into a coherent whole the substance of Euclid XI., the elements of projective geometry, and the elements of descriptive geometry is perhaps a counsel of perfection. Mr. Unwin has confined himself to the subject of descriptive geometry, and has given the simpler standard methods, illustrated by problems on straight lines and planes, and some applications of the index system to topography. An appendix on the elements of three dimensional coordinate geometry must not be overlooked. The suggestion may be offered that the weaker brethren would have found help from a few more hints as to models, or means of constructing or realising the figures dealt with. The book is well printed, and clearly and accurately expressed. C. S. J.

Didaktik des mathematischen Unterrichts. By ALOIS HÖFLER. Bound M. 12. 1910. (B. G. Teubner, Leipzig.)

This volume is the first of a series of ten handbooks on the teaching of mathematics and the natural sciences ("Didaktische Handbücher für den realistischen Unterricht an höheren Schulen") under the editorship of Dr. A. Höfler and Dr. F. Poske. Volumes II. to VII. are to be on various departments of natural science; and the last two volumes are to be by Dr. Höfler, and on "Philosophische

Propädeutik" and "Das Verhältnis der realistischen Unterrichtsfächer zu den sogenannten humanistischen" respectively.

At the present time, when those engaged in teaching, here and in some other countries as well as in Germany, show great activity in studying and reforming methods of teaching, this book should be of the utmost value, written as it is by a philosopher of great repute, and emphasizing as it does, and as one would expect, the great didactic unity of all realistic (realistisch) teaching (p. vii).

In this notice we shall try to abstract from those details of the book which seem to have a bearing merely on the schools of a particular group of nations, and point out what may be called the residuum, which is of quite general moment. This seems all the more necessary as the literature on mathematical didactics which Dr. Höfler quotes is principally of German origin, such as the works of Reidt (1886, 1906) and Simon (1895, 1906, 1908). This almost exclusive reference to German books makes discussions like that for and against Euclid (p. 188) and on Kantianism in mathematics (p. 193) rather incomplete.

Dr. Höfler's book is a detailed and practical treatise on the teaching of arithmetic, geometry, stereometry, algebra, trigonometry, analytical geometry, probabilities, the calculus, etc., to scholars of ages from ten to eighteen years. It is full of relevant criticism, among which we may specially remark that on specimen school-texts, and on the constantly recurring false teaching that a number is an *expression* or a sign (p. 61; cf. pp. 399, 433 *sqq.*).

The most important aspects of Dr. Höfler's book seem to me to be: The emphasis on what seems to be the fact that the most important problems in teaching mathematics are the strengthening of the capacity for spatial intuition, and the accustoming to functional thought (p. 16)—the manner of thinking in terms of functions forming the natural limit between lower and higher mathematics (pp. 18-27; cf. pp. 153, 265, 311)—; and the emphasis on the value in teaching of a study of Galilei's investigations on the law of falling bodies,* in connexion with the beginnings of analytical geometry and the calculus, as well as mechanics (see, *e.g.* pp. 84, 313, 321, 411, 424).

We have already referred to the welcome advance in mathematics made by the rejection of the idea that numbers are signs, and indeed the treatment of the whole question as to whether an attempt should be made to define integer, rational, relative (p. 141) and irrational† numbers, or not seems quite excellent. Other suggestively treated subjects are the inadvisability of arithmetisation when teaching elements (p. 259), and the treatment of the conception of a limit, in connexion with the puzzle of Achilles and the tortoise (pp. 340-345).

To sum up, then, Dr. Höfler seems to us to have written a valuable book which is interesting to everybody, of whatever nation, who is interested in mathematical education; and this in spite of the fact that it refers mainly to the teaching in German schools, and neglects entirely such work as the English and Italian logical researches into mathematics, much of which throws a vivid light on some didactical questions.

PHILIP E. B. JOURDAIN.

Handbuch des mathematischen Unterrichts. By W. KILLING and H. HOVESTADT. Pp. viii + 456. Bound, M. 10. (Teubner, Leipzig.)

It is the opinion of Drs. Killing and Hovestadt that the proposals for a reform of mathematical teaching are usually concerned with questions of method, and the majority of text-books are quite out of touch with the mathematical science of to-day. The present volume is devoted to geometry treated in accordance with the method of Hilbert's well-known work on the foundations of geometry.

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* Of Galilei's *Discorsi* there is a German translation in Ostwald's *Klassiker der exakten Wissenschaften*, and a good account of much of Galilei's work in the English translation of Mach's *Mechanics*.

† That the word "irrational" is merely a bad translation of "ἄλογος," which here means "inexpressible," is remarked on p. 248. This remark seems to have been made first by Kepler in 1649 (according to a note by Terquem in *Journ. de Math.*, t. i., 1836, p. 101).

We think that, when dealing with irrational numbers and limits, Dr. Höfler might with advantage have introduced such a clear discussion of the logical error involved in defining a real number as a limit (in the usual sense) as is found in A. Rösler's tract *Die neueren Definitionsformen der irrationalen Zahlen und ihre Bedeutung für die Schule* (Hannover, 1886).

The book is divided into 24 sections: the first nine deal with the foundations of geometry on the axiomatic method, the tenth is on "geometrical logic," and § 11 to § 24 are principally devoted to school geometry. In this last part (§ 12, pp. 232-242), the authors showed how the modern standpoint of science affords what would seem to be decisive points of view for teaching, and in § 17 (pp. 289-318) they give a discussion of modern theories of irrational numbers.

It seems to us indisputable that certain results of modern mathematics are relevant even to elementary teaching; and this seems to be the case, moreover, with respect to some things not mentioned by Drs. Killing and Hovestadt: for example, with respect to the theory of *definition*. From the point of view of this theory, Frege and others have criticized Hilbert's system,* which certainly, in the form which Hilbert gave it, cannot claim to be more than inferior to the—partly anterior, partly contemporaneous—work of Peano and Pieri.† Also—and this is even more important—Hilbert's method disguises what has, thanks mainly to the work of Russell, shown itself to be the nature of all pure mathematics, including geometry: the statement of implications between premisses and consequents, not of the truth of these premisses or of these consequents.

In elementary teaching, we must, it seems, begin by treating geometry as having for its object the description of an aspect of nature, and afterwards, by considering the growth of non-Euclidean geometry, allow it to appear that, in mathematics, we do not have to deal with an experimental science of space, but with a class of implications of the form: *If* A holds of entries $x, y, z, \ldots$, then B holds of them. Here $x, y, z, \ldots$ are *any* entities, and we have no indefinables such as 'points.' Hilbert's geometry falls between two stools: it is too abstract for elementary teaching, and it does not allow the exclusively implicational and logical character of pure geometry to appear. Moreover, the abstractions do not quite fulfil valuable purposes, since, from a logical point of view, they are open to criticism. And yet, neither the work of Peano, Pieri and Russell in geometry, nor the work of Frege, Peano and Russell in logic, is mentioned in the whole book.

We have expressed our doubts at some length as to the advisability of teaching geometry on the axiomatic method. But let us add that, from § 11 on, the book contains very much of value to a teacher. Thus, on p. 295, Drs. Killing and Hovestadt successfully avoid (after H. Weber) the errors of nominalism and of ascribing creative power to definitions by defining a real number *as a* "Schnitt," and, on p. 303, they avoid the greater temptation to nominalism offered by Cantor's theory of irrationals, by reducing it to Dedekind's theory.

This is a very instructive book for those who wish to instruct themselves and others.

PHILIP E. B. JOURDAIN.

Theories of Parallelism: an Historical Critique. By WILLIAM BARRETT FRANKLAND, M.A., sometime Fellow of Clare College, Cambridge; Vicar of Wrawby. Pp. xviii + 70. 3s. net. 1910. (Cambridge Univ. Press.)

This book contains notes, drawn up in chronological order, of the treatment of parallelism by about forty geometers of rank, among whom the most notable are Euclid, Proclus, Wallis, Leibniz, Saccheri, Lambert, Bertrand, Playfair, Gauss, the two Bolyais, Legendre, Lobatchewski, Meikle, Riemann, Cayley, Helmholtz, Beltrami, Clifford, Klein, and Newcomb. Two notes, on analytical geometry for the metabolic hypothesis, and on planetary motion for this hypothesis, complete the volume, and contain original work.

Detailed historical and critical notes like these on a special subject are all the more valuable, because the subject evidently cannot be treated as fully as it deserves in a general history of mathematics. While we agree with Mr. Frankland when he says (p. vii) that "the historical study of this matter is

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*The "*implicit* definitions" (or "definitions by postulates") so frequently used by Hilbert do not, in general, *define* things, and, though axioms are necessary at the beginning of logic, they have shown themselves to be unnecessary at the beginnings of arithmetic and geometry (cf. Russell, *The Principles of Mathematics*, Cambridge, 1903, pp. v, 4-5, 11, 429-430; Couturat, *Les Principes des Mathématiques*, Paris, 1905, pp. 157-159).

†Cf. Couturat, *Rev. de Métaphys. et de Morale*, xiv., 1906, pp. 210-211.

interesting, not least because it displays, in a manner partly amusing and partly pathetic, the dulnesses, limitations, prejudices and aberrations of professedly logical minds," it seems to us that the chief interest that lies in a study of the theories of parallelism is the perception of the gradual uncovering of the truth that in geometry we have to deal with implications of the form "A implies B," and not with the question as to the truth of A or B. For this reason we should have preferred a rather longer notice to have been given (on p. 44) to Kelland's work. In it there is—for the first time, perhaps, in English—a discussion of the question as to what would follow if the parallel axiom were false.

The bibliographical references might sometimes with advantage be made fuller, but this is hardly a defect worth mentioning in this exceedingly useful book.

PHILIP E. B. JOURDAIN.

Einführung in die Theorie der partiellen Differentialgleichungen. By J. HORN. 10 mks. 1910. (Sammlung Schubert.)

The theory of linear partial differential equations has now become so closely associated with the theory of integral equations that a representative account of the former subject is not complete without some mention of the latter.

The fact that Dr. Horn has devoted the second half of his book to the developments connected with the theory of integral equations will be very welcome to many students.

The book does not contain very much that is new, but there is a clear and faithful account of the work of Fredholm, Hilbert, Picard, and their followers. The account of the work done on the expansions in series of normal functions is not up-to-date, as the important results obtained by Prof. Hobson are not mentioned.

The author gives prominence to the theoretical developments depending on the use of Green's functions. This is quite right from a theoretical point of view and will appeal to the class of mathematicians who regard the establishment of an existence theorem as almost the final word on a subject. Unfortunately, explicit expressions for the Green's functions are difficult to find except in a few simple cases. An alternative method, which is based on the use of various types of definite integral solutions of a partial differential equation, leads to the consideration of integral equations of the first kind.

The study of these equations is very interesting and instructive, but at the same time very difficult. It promises, however, to be of some practical value in enabling us to obtain a solution of the partial differential equation in a finite form.

In some cases, for instance, an integral equation of the first kind can be used to make the solution of a partial differential equation depend on that of an ordinary differential equation involving an arbitrary parameter.

It may be useful to remark that the account of integral equations given in this book covers different ground to that which is covered in R. d'Adhémar's *L'Equation de Fredholm et les problèmes de Dirichlet et de Neumann*, Bôcher's *Introduction to the Study of Integral Equations*, and Kowalevski's *Determinants*.

H. BATEMAN.

Theorie des Potentials und der Kugelfunktionen. Vol. I. By A. WANGERIN. 6 m. 60 pf. 1909. (Sammlung Schubert.)

The first volume of this book is devoted to the theory of the gravitational and electric potential. It contains a clear account of the usual material and also a careful investigation of the behaviour of the different derivatives of the potential function.

Some of these investigations are needed for the derivation of Fredholm's equation and for the method of successive approximations. The care that is needed may be judged from the amount of work that has been done on the subject by Neumann, Liapounoff, Plemelj, Korn, Petrini, and others.

English readers who are interested in the question as to how far the fundamental equations of potential theory have been established, will do well to consult the recent papers of Drs. Bromwich and Leathem in the *Proceedings* of the London Mathematical Society.

H. BATEMAN.

Factor Table for the First Ten Millions. By D. N. LEHMER. Washington, D.C., Carnegie Institution of Washington, 1909. Pp. xiv+476.

The most important factor tables hitherto published are those by Burckhardt for the first three millions, Glaisher for the next three millions, and Dase and Rosenberg for the 7th to 9th millions. In view of the difficulty of obtaining certain of these tables, the appearance of Lehmer's table in a single volume is timely. Since the new table was constructed independently of the earlier tables, it was possible to eliminate errors by noting discrepancies detected by a comparison, made five times, entry for entry. An observation of the rate of error made in the earlier tables shows that there is less than an even chance that both the new table and the earlier tables shall contain an error in the same place in the ten millions. Lastly, a very important check was made by counting the primes within various limits and comparing these counts with the numbers computed by Bertelsen by Meissl's method, the results tallying in every case. It is therefore highly probable that Lehmer's table contains no error in regard to the primes, and it is just here that one using the table is forced to rely entirely upon the accuracy of the table, in contrast with a composite entry and with tables for continuous functions. L. E. DICKSON.

The University, Chicago.

The Public School Geometry. By F. J. W. WHIPPLE, M.A. Pp. xii+154. 1910. J. M. Dent & Sons.

It was somewhat difficult to express a decided opinion upon Mr. Whipple's book for this reason. It is intended for the use of pupils who are quite beginners, and yet, within the limit of 151 pages we find such subjects dealt with as the radii of inscribed and escribed circles, Heron's formula for the area of a triangle, Apollonius' circle, etc. Still, when we study the book and its methods carefully, these results seem attainable in the hands of a good teacher. Mr. Whipple recognises that at an early stage a boy's mental attitude leads him to grasp new truths which are presented to him in a concrete form much more readily than when they are supported by abstract reasoning, and he very wisely makes use of inductive rather than deductive proofs. Another very prominent feature of the book consists in the 2,216 examples. As a rule we do not advocate too many riders, which the ordinary student looks at with wonder, but with very little desire to attempt them because their proofs are almost invariably deductive, and therefore make little or no appeal to his feelings of curiosity. But Mr. Whipple's ingenuity enables him to leave the beaten track, and to give a class of examples which cannot fail to have a stimulating effect upon a boy's mind. From this point of view No. 2104 is a model rider, and the same can be said of many of the early ones.

We are not quite sure that some of Mr. Whipple's new terms appeal to us, *e.g.* stretch=distance or length, way=construction, cutter=transversal, (capital) Π=2 right angles, checks=rectangles, etc. His proofs are of various kinds, viz. inductive, deductive, folding, Euclidean, algebraic, and elastic, of which the last named is certainly novel, and we are not sure whether it is inductive, or what it is. Some of the proofs are excellent, *e.g.* that on p. 60, but we think the geometrico-algebraic proof of Pythagoras' Theorem given on p. 91 would hardly carry conviction to the mind of a beginner. The circle chapter seems rather incomplete, as nothing is given about the equality of angles on equal areas in equal circles, etc., and on the other hand we could have spared the pages dealing with the spherometer and solid geometry, which seem too hard, and somewhat out of place.

On the whole the book is distinctly original, and is evidently written by a teacher who loves his subject and is gifted with an invaluable fund of imagination which he uses to good effect. In the hands of such a teacher, who would always be on the watch carefully to guide his pupils and encourage them to work out the examples, it could not fail to be productive of good results. We can cordially recommend it. JOHN J. MILNE.

Führer durch die Mathematische Litteratur, mit besonderer Berücksichtigung der Historisch-wichtigen Schriften. By F. MULLER. Pp. x+252. 7 m. paper covers. 1909. (Teubner.)

This guide should be found useful to many students and teachers who are

unable to call to their aid expert opinion when they desire to extend their reading in any particular branch of Mathematics. The author does not profess to give anything like a complete bibliography—indeed such a task would need the combined erudition of a syndicate of quite exceptionally well-read experts. But although the book will not be of much use to all, there will no doubt be an audience who will appreciate the opportunity thus afforded them of finding in a few moments a trustworthy list giving the titles, number of pages, date of publication, etc., of books and memoirs upon any branch of the subject in which they may be interested.

Des Notations Mathématiques. Enumération, Choix, et Usage. By D. ANDRÉ. 16 frcs. Pp. xviii+501. 1909. (Gauthier-Villars.)

Considering the importance of the *rôle* played by notation in the development of mathematics, it is somewhat surprising that we have had to wait for a treatise such as this until this year of grace. In his article in the *Penny Cyclopaedia* on Symbols and Notations, De Morgan complained that nothing at all had been written in modern times on the subject except Babbage's article in the *Edinburgh Encyclopaedia*. Of course there is a good deal of material scattered here and there through the pages of mathematical writers, especially where a new notation has been invented, and the author has deemed it necessary to offer some explanation or justification of the course he has adopted. But in the large majority of cases, writers have adopted notations involving novelties without condescending to give their reasons. Inappropriate symbols disappear of course in the long run, but their use implies at any rate a temporary check to progress. And just as some authors have almost a genius for clothing their ideas in symbolism that is both ambiguous and forbidding, so others, again, are equally notable for the felicity with which they devise symbolism that is absolutely flawless and easily assimilated. Lagrange is the classical instance of the perfect elegance of analytical style. Elie de Beaumont, in his *Eloge de Legendre*, might seem to be on the verge of rhapsody when he wrote, "Pures et faciles comme les vers de Racine, les formules de Lagrange ont augmenté le nombre des adeptes de la Science, en même temps qu'elles ont facilités leurs travaux." But Biot has put the case in unmistakable form: "Lorsqu'on a une fois lu un Mémoire de Lagrange, on n'a jamais besoin d'y revenir; on a tout compris, et pour toujours." There can be no higher praise than this. The subject of notation is, of course, of very great interest from the purely historical side, and it must not be forgotten that no one is more concerned than the teacher in the employment of simple forms to denote simple notions. De Morgan has somewhere referred to the lack of economy in using long words to denote what is of frequent occurrence, and he contrasts in this connection the words "numerator," "denominator," with "cusp." And when we consider the unnecessary pitfalls with which we are familiar, *e.g.* the double use of the word "tangent," or the obliquity which so perversely refuses to use the simple and immediately intelligible form $\arcsin x$ for the utterly unjustifiable $\sin^{-1}x$, we have said enough to show that in choice of language and of symbols alike the science is far from impeccable. One more point. De Morgan rightly complained that in the printing of mathematics far too much is left to the compositor. He looked for no improvement until some printer should take to mathematics as his recreation, or some mathematician embrace the printer's craft as a hobby. The various difficulties that arise in the printing of mathematical material are faithfully dealt with by M. André. The Council of the Royal Society has recently issued a short memorandum on the best way to avoid the mistakes that are likely to occur—mistakes, we mean, which are largely due to technical causes over which neither the compositor nor the reader of proofs has in reality any final control at all. We hope in the near future to print this document, which was in the first instance drawn up by Sir Joseph Larmor. The essential portions of it are to be found in *Nature* (Mar. 15th, 1909). No apology is needed for this proposal, as it occasionally happens that an editor has practically to rewrite the whole of a paper sent in for publication. Space forbids a detailed analysis of M. André's well-conceived volume, but we have not the slightest doubt that when its merits become known it will find a place on the shelves of all who are interested in an important branch of the history of mathematics, who wish to cultivate a taste for

mathematical style, and who realise what is implied in the phrase from Stoll which M. André has taken as a motto for this book: "*de quibusdam magni momenti minutiis.*"

Problèmes et Exercices de Mathématiques Générales. By E. FABRY. Pp. 420. 10 frcs. 1910. (Hermann, Paris.)

Courses of instruction in higher mathematics for the benefit of future engineers, chemists, etc., have for some few years been in full swing at most of the French Universities. Prof. Fabry, who has already published a text-book to meet the needs of such students, entitled *Traité de Mathématiques Générales*, has now thought it advisable to supplement that volume by a collection of exercises and problems on the principal parts of the course. Hence the volume under notice, which contains questions set in four sections—Algebra, Analytical Geometry, Analysis, and Mechanics. They are 738 in number, and the second part contains full solutions. The teacher may find the collection useful, and the private student who has wit enough to use solutions with discretion should, we think, find the book a valuable aid in his preparation for such portions of British Honours examinations as are here covered.

A New Algebra. By S. BARNARD and J. M. CHILD. Parts I.-IV., with answers. Pp. x+534. 4s. 1909. (Macmillan.)

The book before us is one that no teacher can afford to ignore. It is unique of its kind, being the only school algebra to our knowledge in which we have at one and the same time a sound treatment for the purposes of the young of the modern idea of number and a class text-book covering all the ground required for the beginner. It may be thought that the development of the subject on the lines selected by the authors must necessarily be forbidding, and present considerable difficulties to those whose only acquaintance with number is that derived from their measurements. We are inclined to think that the difficulties have been somewhat exaggerated. The method of presentation may be made attractive, and the judicious selection of the stage at which the notions of negative, fractional, and irrational number are introduced will go far to convince the teacher that the experiment so successfully made by the authors in this volume is perfectly feasible. We mention the importance of the stage at which the idea of negative number, for instance, is introduced, because Messrs. Barnard and Child have come to the conclusion that if the recommendations of the Mathematical Association Report on the point were followed, it was not easy to make the earlier exercises of the variety that is desirable. That is an important practical issue, though the matter is one that will naturally settle itself in the course of time. We have no doubt that many will avail themselves of the columns of the *Gazette* for the purpose of recording their views as soon as they begin to be shaped by experience. It is just for such an object as this that the *Gazette* exists. And so also there will in time be to hand plenty of instances of the ingenuity of teachers in devising instructive methods of presentation. For instance, instead of at once *inventing* new symbols, $0, -1, -2, \ldots$, we might illustrate as follows. Arrange the natural or counting numbers in ascending order from left to right. Imagine a sheet of looking-glass placed as far to the left of the "1" as the "1" is from the "2," the "2" from the "3," etc. On looking into the glass the boy will see a row of images in precisely the same relative order as the counting numbers. If we wish to add 2 and 7 we may start with the 2 and count 7 places to the *right*, or with the 7 and count 2 places to the *right*. For convenience we may write a minus sign over each of the phantoms in the glass. To add -2 and 3 we start from the -2 and move 3 places to the *right*, and so on. The exceptional position of zero in *both* the series is noted. In subtraction we move so many places to the *left*. We have found this way of looking at the negative numbers amusing to children, and serve very well in breaking gently to them the new idea. But directly the attention of a few hundred teachers is directed to the points, though at first it may be a case of *tot homines tot sententiae*, we have no doubt it will not be long before there will be a general consensus as to the simplest method of approach. Apart from the notion of number, it will be found that Messrs. Barnard and Child have produced an excellent text-book, which is well worth a trial. Such a book had to come sooner or later, and teachers may well congratulate themselves that the pioneer volume is from such accomplished hands.

An Elementary Treatise on the Dynamics of a Particle and of Rigid Bodies. By S. L. LONEY. Pp. viii+374. 1909. (Cambridge University Press.)

Mr. Loney's name is too well known as a writer of text-books for it to be necessary to do more than call attention to the scope of the class-book he now publishes upon Particle and Rigid Dynamics. The student is supposed to have read some course of elementary Dynamics, to have a fair working knowledge of the Differential and Integral Calculus, and, with the aid of the Differential Equations solved in the text and appendix, he will have all the equipment necessary for his purposes. The ground covered is that required for a Science or Engineering Degree in Applied Mathematics, or for Junior Students for an Honours Degree in Mathematics. The work is clearly that of a skilful and experienced teacher, and candidates for the examinations mentioned will probably find in it all that they require in order to cut a respectable figure when their time of trial arrives. On the other hand, we cannot look upon the book as doing much more than take the student through a subject for examination purposes, however excellently this may be done. As a treatise on cut and dried lines, it has the qualities that have made its more elementary predecessors of value in the past. But we think the author would have made his book far more inspiring and attractive had he remembered that the age at which students are fit to begin this stage of their reading is also the age at which it is well to cultivate their imaginative capacity. In another couple of years every boy who reaches this stage will have had his ideas developed largely through the experimental course, which is everywhere forming an essential part of the preliminary training in applied mathematics. This then is the time for a more critical examination of the familiar concepts, for a glance, or even more than a glance, at the history of the subject and the development of principles. From the point of view of the student's mental growth, it does not seem to us to be right that at 18 or 19 he should be left entirely to himself to discover the existence of a working hypothesis where he has hitherto met the infallible rigour of a law. We cannot, however, blame the author for not transgressing the limits he has deliberately laid down for himself in this otherwise excellent manual, and, of course, we are open to challenge as to our view of the propriety of allowing a student to suppose that the problems dealt with in a mathematical text-book are exactly those we find in the world around us.

Elementary Mechanics of Solids and Fluids. By A. C. JONES, M.A., Ph.D., and C. H. BLOMFIELD, M.A., B.Sc. Pp. vi+366+xv. 1909. (London: E. Arnold.)

This book contains abundant and varied examples, and some excellent diagrams (drawn by Mrs. Clement Jones) of the kind that a student may reasonably be expected to reproduce—large, bold, and cleared of trivialities.

That hydrostatics has been postponed too long by the mathematical side, and dealt with too early by the physical side, of most schools is reasonably certain; and full credit is due to the authors for their attempt to preserve what is best in the traditional teaching of mechanics, and to coordinate their work with "elementary general science," in which the properties of fluids play a leading part. This attempt, however, involves difficulties which in the opinion of the present reviewer have not been fully overcome. Perhaps the fairest plan is to set out the authors' sequence of topics, viz.: Frame of reference, velocity, acceleration, mass, density (the material being tacitly assumed homogeneous), resultant velocity, relative velocity, Newton's second law, pressure in a fluid, Lami's theorem, Bow's method, Boyle's law, resolution of forces, moments, parallel forces, levers, conditions of equilibrium, air and water pumps, centre of gravity, floating bodies, Bow's method again, whole pressure, projectiles, motion in a circle, work, energy, pullies, impact.

The personal opinion must be expressed that this order cannot be defended either on scientific or pedagogic grounds, and in fact friction, which is discussed at p. 275, creeps in at p. 58.

However, as all readers of Dr. Jones' *Notes on Analytical Geometry* will be prepared to expect, there are many valuable and instructive remarks and examples throughout the book.

A Treatise on Spherical Astronomy. By Sir Robert Ball, F.R.S. Pp. xii+506. 12s. net. 1910. (Cambridge University Press.)

"By Spherical Astronomy I mean that part of Mathematical Astronomy which lies between the vast domain of Dynamical Astronomy on the one hand and the multitudinous details of Practical Astronomy on the other."

These are the limits which Sir Robert Ball has set himself in the volume before us, and with the skill we expect in a writer and teacher of his long experience he has produced, from the immense amount of detail at his disposal, a text-book which not merely satisfies the requirements of Tripos candidates and the like, but which takes its place among the classics dealing with this branch of the science of which the author is so distinguished an exponent. Fortunately, for the reader and the student, Sir Robert has been tempted more than once outside the bounds, and on each occasion with definite intent and to some purpose. In his last two chapters he crosses the frontier of Practical Astronomy, and sets forth the fundamental geometrical principles underlying all astronomical instruments. He considers a "generalised instrument," a geometrical abstraction, in the theory of which the principles of ordinary instruments fall into line as special cases. Here we find in its proper place the novel and useful note contributed to the *Monthly Notices* for Jan. 1908, in which is derived the fundamental equation which can be made to apply to the altazimuth, the meridian circle, the prime vertical instrument, and the almucantar. The fundamental instruments themselves are treated in the final sections of the book. The intrusion into the field of dynamical astronomy is found in the chapter dealing with Kepler's and Newton's laws and their applications. Sir Robert is also to be congratulated on the felicitous introduction of the term *nole* to denote "that pole of a graduated great circle which lies towards the left hand of a man walking on the outside of the sphere along the circle in the direction in which the graduation increases," with, of course, "anti-nole" to denote the opposite pole. The archaic "nole" is still to be heard in remote parts of the country, in the sense of head, or neck, or even navel. The selection of nole from among the various forms in which the word appears is singularly happy as suggesting "north pole." The collection of examples includes a large number taken from College and Tripos papers. Solutions are in many cases given, and some of them are extremely neat. The book fills a gap. The student need no longer apply to American or German sources for the course required for Tripos purposes, and, at the same time, the book is distinct in character from such a work as Newcomb's "Compendium." Sections which may be omitted on a first reading are marked with an asterisk. What is left forms as admirable an introduction to the mathematics of Spherical Astronomy as exists in any language.

Elements of the Differential and Integral Calculus. By A. E. H. Love, F.R.S. 5s. 1909. (Cambridge University Press.)

"The Principles of the Differential and Integral Calculus ought to be counted as a part of the intellectual heritage of every educated man or woman in the twentieth century no less than the Copernican system or the Darwinian theory." It is half a century since the then Sedleian Professor of Natural Philosophy in the University of Oxford published the first of four volumes dealing with the Infinitesimal Calculus. How Bartholomew Price would have rubbed his eyes could he have looked into the future and read the above sentence from the preface to a work on the Calculus by a successor in his Chair! For in his time he could say of pure and applied mathematics that "these, indeed, require very heavy and arduous study, inasmuch as they have peculiar nomenclature, language, and processes, and thus it is only to the few generally who have made them their particular study that they offer great interest." For the last six years Prof. Love has been giving an annual course of lectures on the elements of the Differential and Integral Calculus to classes mainly of chemists and engineers. He has necessarily arrived at a very clear notion of the amount of mathematical attainment that is requisite as a preliminary equipment. He finds, for example, that the Binomial Theorem and the Addition Equation in Trigonometry "are quite unnecessary." And the book before us demands no previous knowledge of mathematics "beyond the most elementary notions of geometry, a little algebra, including the law of indices, and the definition of the trigonometric functions."

The book, in fact, "is intended merely to help the reader to make a beginning." Within the limits thus laid down Prof. Love has achieved a distinct success. Nothing could well be simpler than the exposition, and the arrangement is all that can be desired. In a word, it is admirably done.

Serret, Lehrbuch der Differential- u. Integralrechnung. Vol. III., 3rd edition. Edited by G. SCHEFFERS. **Differentialgleichungen und Variationsrechnung.** Pp. xii+657. 13 marks. 1909. (Teubner.)

Dr. Scheffers has now completely rewritten and brought up to date Serret's work on the Calculus. Of the present volume eleven-twelfths are devoted to Differential Equations, and the rest, a single chapter, treats of the Calculus of variations.

Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung. By F. DINGELDEY. Vol. I. *Differentialrechnung.* Pp. iv+202. 6 m. bound. 1910. (Teubner.)

This useful collection of examples is divided into chapters under the headings that are usually found in text-books on the Differential Calculus. Each chapter begins with a précis of the subject-matter, and then follow examples, often with ample indications of the method of solution. The applications are made in the fields of geometry, chemistry and physics, engineering, etc., and should be found a useful supplement to the sets given in books on practical mathematics. In the 200 pages are about 100 diagrams. The book is beautifully printed and well bound. It forms the 32nd volume of the valuable series issued by Messrs. Teubner entitled, *Sammlung von Lehrbuchern auf dem Gebiete der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen.*

Analytische Geometrie der Kegelschnitte. By W. DETTE. Pp. vi+232. Bound, 4.40 m. 1909. (Teubner.)

Nearly one-third of this book is devoted to solutions, hints, and answers to an excellent, carefully chosen and graded set of questions on the subject-matter of the various chapters. The arrangement in the text differs somewhat from that to which teachers are accustomed. For instance the treatment of the circle precedes that of the straight line. Emphasis is laid on the principle of duality from the outset. The ellipse follows the line, and the parabola is defined as the limit of a series of ellipses. The theory of poles and polars is lucidly developed and upon it is made to depend the theories of centres, diameters, and directrices. In all cases the canonical forms of the equations of the conics are obtained before the general equation is attacked. The questions for solution follow the text and are arranged in 46 sets averaging about 15-20 in each set. The book is beautifully printed, and is certainly worthy the attention of the student, and of the teacher to whom elasticity in methods is no bugbear.

Analytical Geometry of the Conic Sections. By E. H. ASKWITH. Pp. xiv+443. 7s. 6d. net. (A. & C. Black.)

We hope that by the time this belated notice of Dr. Askwith's admirable treatise appears the merits of the book will have been substantially recognised both by private students and by teachers. There are several reasons why we do not think the book can be recommended for beginners, save perhaps for boys of exceptional capacity. The text demands a greater readiness in the manipulation of algebraical expressions than is likely to be possessed by a boy who is attacking the subject for the first time. The exercises are not easy enough for the beginner, and would require considerable additions in the way of supplementary questions, to make them suit the needs of those to whom the subject is new. But for those who have already made a successful attempt to grasp the elements of the subject the chapters i.-xiii. form an almost ideal group for careful revision, and are full of the most instructive and attractively presented material. There are plenty of novelties that will give the teacher, however experienced, food for thought and stimulus to experiment and divergence from the ordinary routine. We might, however, suggest that adherence to the purely analytical treatment throughout may tend to make the handling of material too mechanical. A more frequent reference to the figure gives a firmer hold of principles involved and to

many temperaments will bring to life what may seem at first but dry bones. In the case of loci it is quite possible that the full significance of the results may not be grasped without a geometrical investigation. But it is to the last six chapters of the book that teachers will look with the most interest. No text-book gives us such a survey of homogeneous coordinates and their applications. After carefully studying the five-and-thirty pages devoted to areals as the simplest system of homogeneous coordinates, the student is well equipped for the study of those coordinates in general. Trilinears fall into their natural place and no longer hold in our text-books the somewhat illogical position that they have been allowed to assume in the past. The sections on cross ratios, harmonic section and involution are followed by a chapter on invariants, which is excellent in every detail. Tangential equations and envelopes fill another twenty pages, and the final sections of the book form a model chapter on covariants. We must not omit to draw attention to such sections as 300 (a), which indicates the care to be exercised in dealing with the line at infinity. These betray the practised teacher who is fully aware of the pitfalls that await the average pupil. The splendid collection of examples contains a large number of Scholarship, College, and Tripos questions, and in many cases suggestive hints are given which will be found of value in the individual problem and in those of the same or similar type. In spite of the full table of contents it seems to us, considering the novelty of treatment and the character of the last few chapters, that the author will do well in a future edition to add an index. The pleasure of glancing again and again through Dr. Askwith's pages somewhat atones for its absence, but that feeling does not prevent us from pointing out the inconvenience to those who want to find things in a hurry. No volume on Analytical Geometry can yet avoid a direct challenge of comparison with the great classic of the subject. But the young enthusiast who is touched by the mathematical spirit, and the teacher whose views on methodology are in a state of flux. are not to be pitied as long as they have on their shelves copies of Salmon, Fiedler-Salmon, Scott, and Askwith.

Practical Curve Tracing, with Chapters on Differentiation and Integration. By R. H. DUNCAN. Pp. vii+137. 5s. net. 1910. (Longmans, Green.)

Mr. Duncan's new contribution to the literature of Curve Tracing is intended primarily for the student who has the minimum of equipment of the "mental gymnastic" type of mathematics. The ninth page opens with the discussion of the simple equation of the first degree, and $y=ax+b$ (the relation between x and y being described as "the straight line law") is illustrated by six figures, three arising from the varying of a while b remains constant, and the rest from the variation of b while a remains constant. The line that is drawn as evenly as possible between the various points plotted out is termed the "mean straight line," and the use of a stretched thread is recommended as more satisfactory for practical purposes than that of the set square or ruler. The reader is naturally interested to see how this same method works out when applied to members of the parabolic family. After carefully studying chapters iv.-vi. we cannot help feeling that Mr. Duncan's proposals are pedagogically sound, and we have more than a suspicion that the student who has mastered these chapters will have acquired a more thorough and more permanent grasp of what the term "parameter" implies than he can get, at any rate so rapidly, in any other way. The next two chapters are given up respectively to the exponential family and to the sine curves, simple, compound, and damped. Half a dozen pages suffice for the graphic solution of equations, and the rest of the 116 pages deal with "The slope of a curve—Differentiation" and "The Area of a Curve—Integration." The small collection of examples provided is ample for the author's purpose, and at the end of the book the necessary tables are appended. The only proof of the pudding is of course in the eating. It would be interesting to hear from such of our readers as may be tempted to give Mr. Duncan's methods a trial what their experience of the result may be. Bearing in mind that his book is written for those who are weak in "purely academic Mathematics" there is ample opportunity throughout the book of abbreviating the steps. In many cases, no doubt, the pupil's own resources may be equal to the task, especially if he knows he is invited to suggest "improvements."

Practical Arithmetic for Schools. By W. G. Borchardt. Pp. viii+445 +lxxvi. 3s. 6d.; with Answers 4s. 6d. Part I. 2s.; with Answers 2s. 6d. Part II. same prices. Examples, separately, with or without Answers, 3s.; or Part I. 1s. 6d., Part II. 2s. each, with or without Answers. 1909. (Rivingtons.)

After using for a term Mr. Borchardt's contribution to the already considerable number of text-books on the market, we have come to the conclusion that every teacher should possess a copy whether he is able to introduce it into his classes or not. Having said so much, it would seem to be unnecessary to say any more about its merits.

Savants du Jour. Henri Poincaré; Gaston Darboux. By E. Lebon. Pp. viii+80 and pp. viii+72. 1909-1910. 7 frcs. each. (Gauthier-Villars.)

That there are certain disadvantages inherent in biographies of living persons is obvious enough, and yet it is well that the world should know something of its greatest men while they are yet with them. The man in the street has his needs in this respect amply catered for by paragraphs in the daily press or by magazine articles. But this is not sufficient for those who are themselves interested professionally or otherwise in the particular work that is the preoccupation of the object of their admiration, or of those who are endeavouring humbly to follow in his footsteps. To M. Lebon is due the excellent idea that is embodied in the series called *Savants du Jour*, the two first numbers of which lie before us. It is a far cry from the pemmican that is offered by "Who's Who" to the mass of sometimes quite irrelevant matter which is to be found in the ordinary biography. Neither is entirely satisfactory to the student or to the worker in the same field as the expert as to whose labours information is required. Hence the *raison d'être* of these monographs, which we owe to the energy and enthusiasm of M. Lebon. He assumes that the particular class for which he is providing is desirous to know the nature and the scope of the researches already accomplished by an individual, their relation to work done in the same field or fields by others, and, if possible, what is thought by other experts of the value of the work achieved. And if this is accompanied by an analytic bibliography this class is content. Now this is exactly what M. Lebon does, and gives us in each instance an admirable heliogravure in addition. In the case of the first monograph, which naturally is devoted to M. Henri Poincaré, the purely biographical part is supplied by the historian Frederic Masson, whose duty it was to pronounce the usual panegyric upon the leader of the French scientific world on the day when he became one of the famous "forty." Herein are felicitously sketched the salient features of the career of the subject of this memoir, with a literary skill which is only rivalled in the admirable criticism and analysis of Poincaré's *Science et Méthode* by Emile Faguet, another famous Academician. Of Poincaré's work in Mathematical Physics and in Pure Mathematics we have an excellent summary in the reports presented by M. Gustave Rados to the Hungarian Academy of Science on the occasion of the award of the Bolyai prize. Here is deftly sketched the winner's work in the theory of functions, in Analysis Situs, and in the theory of numbers. At the same time the relation of his labours to those of Cauchy, Lie, Hermite and others is lucidly indicated. As for Poincaré's achievements in the realm of celestial mechanics, we are proud to see that Prof. Sir G. H. Darwin has been selected to give his testimony to the great merits of the famous Frenchman. In the speech delivered * when Poincaré was presented with the gold medal of our Royal Astronomical Society, Sir George Darwin had an opportunity, of which he was not slow to avail himself, of alluding to the "revelation" afforded by the researches of the subject of this monograph in celestial mechanics. After each of these addresses we find a careful bibliography set out under suitable headings—a most convenient arrangement for ready reference to the work of so many-sided a man. The sum total of memoirs, articles, official reports, orations, prefaces,—nothing seems forgotten —amounts to 436 items.

The second volume of the series is devoted to M. Gaston Darboux, and is carried out on parallel lines to those of its predecessor. Perhaps the name

* The date on p. 37 should be 1900. It is correctly given on p. 40.

of M. Darboux will be less familiar to the general public than it is to mathematicians. Readers of the *Gazette* will remember his paper on the development of geometrical methods which was translated for these columns some years ago. But it is not only in methodology that he has made his mark. It has been said that the Chair of Geometry, in which he succeeded Chasles, should now, after twenty years of Darboux, be rather called the Chair of Infinitesimal Geometry. Among the addresses in this monograph are reports by Camille Jordan, Chasles, Hoüel, and Gilbert, from which in turn may be gathered the importance attached by the contemporaries of Darboux to his work in such varied subjects of research as the theory of differential equations, systems of orthogonal surfaces, cyclics, the theory of imaginaries, and analytical and celestial mechanics, etc. We note that M. Lebon has received a grant towards his enterprise from the *Académie des Sciences*. This reminds us of Sterne, when he remarks: "They order, said I, this matter better in France." Let us hope that our official representative scientific bodies are not of opinion that we in this country have no *Savants du Jour*, and that if approached on the subject they will not venture to assert that our student world has "no use" for such monographs as these. On the other hand, there is one matter we order better in England. "The Collected Papers" of our great mathematicians are not so long in making their appearance as are those of their illustrious French confrères. Which also reminds us that we may see a completed edition of Euler within the next decade.

INTERNATIONAL CONGRESS OF MATHEMATICIANS, CAMBRIDGE, 1912.

ADVISORY COMMITTEE TO THE BOARD OF EDUCATION.

Memorandum.

AT the Fourth International Congress of Mathematicians in 1908, resolutions were passed leading to the formation of an International Commission, and subsequently of National Sub-commissions, to present a report on Mathematical Education in different countries to the Fifth International Congress which is to be held at Cambridge in 1912.

The President of the Board of Education has appointed an Advisory Committee which will also act as the British Sub-commission to the International Commission on Mathematical Teaching.

The Board of Education have undertaken to publish papers approved by the Advisory Committee, as publications of the Board's Department of Special Inquiries and Reports of which Dr. H. F. Heath is Director, and it is intended that a limited number of copies of the collected papers shall be placed at the disposal of the International Commission for presentation to the Congress at Cambridge.

BOOKS, ETC., RECEIVED.

The Mathematics Teacher. Vol. II. No. 3. A Quarterly published by the Association of Teachers of Mathematics for the Middle States and Maryland.

Some Common Errors in Elementary Algebra. Approximate Values of π. W. H. SHERK. *Some Suggestions in the Teaching of Geometry.* I. J. SCHWATT. *Mathematics for Service.* E. R. VON NARDROFF.

On the Series for calculating Euler's Constant and the Constant in Stirling's Theorem. By K. J. SANJANA. (Reprint from *Proc. Edin. Math. Soc.*, xxviii. 1909-1910.)

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